

# Morita equivalence classes of 2-blocks of defect three

Charles W. Eaton

22nd September 2014

## Abstract

We give a complete description of the Morita equivalence classes of blocks with elementary abelian defect groups of order 8 and of the derived equivalences between them. A consequence is the verification of Broué's abelian defect group conjecture for these blocks. It also completes the classification of Morita and derived equivalence classes of 2-blocks of defect at most three defined over a suitable field.

## 1 Introduction

Throughout let  $k$  be an algebraically closed field of prime characteristic  $\ell$  and let  $\mathcal{O}$  be a discrete valuation ring with residue field  $k$  and field of fractions  $K$  of characteristic zero. We assume that  $K$  is large enough for the groups under consideration. We consider blocks  $B$  of  $\mathcal{O}G$  with defect group  $D$ .

We are concerned with the description of the Morita and derived equivalence classes of (module categories for) blocks of finite groups with a given defect group  $D$ . We briefly review progress on this problem to date. If  $D$  is an abelian  $p$ -group whose automorphism group is a  $p$ -group, then any block with defect group  $D$  must be nilpotent and so Morita equivalent to  $\mathcal{O}D$  (see [14] and [22]). There are many other examples of  $p$ -groups for which it has been proved that every fusion system is nilpotent, but we do not list these here. If  $D$  is cyclic, then the Morita equivalence classes can be characterised in terms of Brauer trees, in work going back to Brauer and Dade (see [20]). In a series of papers Erdmann characterises the Morita equivalence classes of tame blocks defined over  $k$  except when  $D$  is generalised quaternion and  $B$  has two simple modules (see [8]), although the problem remains open for blocks defined over  $\mathcal{O}$ . The (three) Morita equivalence classes of blocks defined over  $\mathcal{O}$  with defect group  $C_2 \times C_2$  are determined in [19]. When  $D = \langle x, y : x^{2^r} = y^{2^s} = [x, y]^2 = [x, [x, y]] = [y, [x, y]] = 1 \rangle$ , where  $r \geq s \geq 1$  (nonmetacyclic minimal nonabelian 2-group), the Morita equivalence classes of blocks are determined in [24] and [7]. When  $D$  is a homocyclic 2-group, the Morita equivalence classes of blocks are determined in [6].

In this paper we use the classification given in [6] to completely determine the Morita and derived equivalence classes of blocks defined over  $\mathcal{O}$  with defect group  $D \cong C_2 \times C_2 \times C_2$ . As a consequence it follows that Broué's abelian defect group conjecture holds for blocks of defect three. We also note that this completes the classification of Morita equivalence classes of 2-blocks of defect at most three, for

blocks defined over  $k$ . Blocks with elementary abelian defect groups of order 8 have already been studied in [10], where it is shown that Alperin's weight conjecture and the isotypy version of Broué's abelian defect group conjecture hold for these blocks. The results of [10] are needed here, in particular to achieve Morita equivalences over  $\mathcal{O}$  rather than  $k$ .

Before stating the main theorem, we recall the definition of the inertial quotient of  $B$ . Let  $b_D$  be a block of  $\mathcal{O}DC_G(D)$  with Brauer correspondent  $B$ , and write  $N_G(D, b_D)$  for the stabilizer in  $N_G(D)$  of  $b_D$  under conjugation. Then the *inertial quotient* of  $B$  is  $E = N_G(D, b_D)/DC_G(D)$ , an  $\ell'$ -group unique up to isomorphism.

**Theorem 1.1** *Let  $B$  be a block of  $\mathcal{O}G$ , where  $G$  is a finite group. If  $B$  has defect group  $D$  isomorphic to  $C_2 \times C_2 \times C_2$ , then  $B$  is Morita equivalent to the principal block of precisely one of the following:*

- (i)  $D$ ;
- (ii)  $D \rtimes C_3$ ;
- (iii)  $C_2 \times A_5$ , and the inertial quotient is  $C_3$ .
- (iv)  $D \rtimes C_7$ ;
- (v)  $SL_2(8)$ , and the inertial quotient is  $C_7$ ;
- (vi)  $D \rtimes (C_7 \rtimes C_3)$ ;
- (vii)  $J_1$ , and the inertial quotient is  $C_7 \rtimes C_3$ ;
- (viii)  ${}^2G_2(3) \cong \text{Aut}(SL_2(8))$ , and the inertial quotient is  $C_7 \rtimes C_3$ ;

*Blocks are derived equivalent if and only if they have the same inertial quotient.*

A block with defect group  $C_2 \times C_2 \times C_2$  cannot be Morita equivalent to a block with non-isomorphic defect group. This is since Morita equivalence preserves defect and (i) 2-blocks of defect three with abelian defect groups other than  $C_2 \times C_2 \times C_2$  must be nilpotent (and so Morita equivalent to the group algebra of a defect group), (ii) 2-blocks of defect three with nonabelian defect groups have five irreducible characters (whilst the number is eight for blocks with defect group  $C_2 \times C_2 \times C_2$ ).

**Corollary 1.2** *Broué's abelian defect group conjecture holds for all 2-blocks with defect at most three. That is, let  $B$  be a block of  $\mathcal{O}G$  for a finite group  $G$  with defect group  $D$  of order dividing 8, and let  $b$  be the unique block of  $\mathcal{O}N_G(D)$  with Brauer correspondent  $B$ . Then  $B$  and  $b$  have derived equivalent module categories.*

PROOF. If a defect group  $D$  are isomorphic to  $C_2$ ,  $C_4$ ,  $C_4 \times C_2$  or  $C_8$ , then the block is nilpotent, in which case the conjecture holds automatically since  $\text{Aut}(D)$  is a 2-group. If  $D \cong C_2 \times C_2$ , then the result follows from [19]. Suppose that  $D \cong C_2 \times C_2 \times C_2$ . By Theorem 1.1 the derived equivalence class of  $B$  is uniquely determined by the number  $l(B)$  of irreducible Brauer characters. Since every block with defect group  $D$  has eight irreducible characters, it is a consequence of Brauer's second main theorem that  $l(B) = l(b)$  and the result follows.  $\square$

Note that we do not prove that there are splendid derived equivalences of blocks.

**Corollary 1.3** *Let  $B$  be a block with defect group  $D \cong C_2 \times C_2 \times C_2$ . Then  $B$  has Loewy length  $LL(B)$  equal to 4, 6 or 7.*

PROOF. By Theorem 1.1 it suffices to consider cases (i)-(viii) in the notation of that theorem. In cases (i), (ii), (iv) and (vi), where  $D \triangleleft G$  and  $[G : D]$  is odd, we have that  $LL(B) = LL(kD) = 4$ , by [12, 4.1]. In case (iii)  $LL(B) = 6$ . In the remaining cases  $LL(B) = 7$  by [1] and [18], again using [12, 4.1].  $\square$

**Corollary 1.4** *Let  $B$  be a 2-block of defect at most 3, then the Cartan invariants of  $B$  are at most the order of a defect group.*

Of course the above does not hold in generality.

Since we now have a complete list of Cartan matrices (up to ordering of the simple modules), and indeed the decomposition matrices, for 2-blocks of defect at most 3, it would be interesting to look for possible concrete restrictions on Cartan matrices.

## 2 Quoted results

The following proposition will be used when considering automorphism groups of simple groups. It gathers together two propositions from [10], which in turn gathers results from [5] and [15].

**Proposition 2.1** *Let  $\ell$  be any prime and let  $G$  be a finite group and  $N \triangleleft G$  with  $[G : N] = w$  a prime not equal to  $\ell$ . Let  $b$  be a  $G$ -stable  $\ell$ -block of  $\mathcal{O}N$ . Then either each block of  $\mathcal{O}G$  covering  $b$  is Morita equivalent to  $b$ , or there is a unique block of  $\mathcal{O}G$  covering  $b$ . In the former case,  $B$  and  $b$  have isomorphic inertial quotient.*

PROOF. Note that the group  $G[b]$  of elements of  $G$  acting as inner automorphisms on  $b$  is a normal subgroup of  $G$  containing  $N$ . If  $G[b] = G$ , then each block of  $G$  covering  $b$  is source algebra equivalent to  $b$  by [10, 2.2], and has inertial quotient isomorphic to that of  $b$  by [10, 3.4]. If  $G[b] = N$ , then there is a unique block of  $G$  covering  $b$  by [10, 2.3].  $\square$

The following is a distillation of those results in [16] which are relevant here.

**Proposition 2.2** ([16]) *Let  $G$  be a finite group and  $N \triangleleft G$ . Let  $B$  be a block of  $\mathcal{O}G$  with defect group  $D$  covering a  $G$ -stable nilpotent block  $b$  of  $\mathcal{O}N$  with defect group  $D \cap N$ . Then there is a finite group  $L$  and  $M \triangleleft L$  such that (i)  $M \cong D \cap N$ , (ii)  $L/M \cong G/N$ , (iii) there is a subgroup  $D_L$  of  $L$  with  $D_L \cong D$  and  $D_L \cap M \cong D \cap N$ , and (iv) there is a central extension  $\tilde{L}$  of  $L$  by an  $\ell'$ -group, and a block  $\tilde{B}$  of  $\mathcal{O}\tilde{L}$  which is Morita equivalent to  $B$  and has defect group  $\tilde{D} \cong D$ .*

**Proposition 2.3** ([26]) *Let  $B$  be an  $\ell$ -block of  $\mathcal{O}G$  for a finite group  $G$  and let  $Z \leq O_\ell(Z(G))$ . Let  $\bar{B}$  be the unique block of  $\mathcal{O}(G/Z)$  corresponding to  $B$ . Then  $B$  is nilpotent if and only if  $\bar{B}$  is nilpotent.*

**Proposition 2.4** ([6]) *Let  $B$  be a block of  $\mathcal{O}G$  for a quasisimple group  $G$  with elementary abelian defect group  $D$  of order 8. Then one of the following occurs:*

- (i)  $G \cong SL_2(8)$  and  $B$  is the principal block;

- (ii)  $G \cong {}^2G_2(q)$ , where  $q = 3^{2m+1}$  for some  $m \in \mathbb{N}$ , and  $B$  is the principal block;
- (iii)  $G \cong J_1$  and  $B$  is the principal block;
- (iv)  $G \cong Co_3$  and  $B$  is the unique non-principal 2-block of defect 3;
- (v)  $G$  is of type  $D_n(q)$  or  $E_7(q)$  for some  $q$  of odd prime power order,  $O_2(G) = 1$  and  $B$  is Morita equivalent to the principal block of  $C_2 \times A_5$  or of  $C_2 \times A_4$ .
- (vi)  $|O_2(G)| = 2$  and  $D/O_2(G)$  is a Klein four group;
- (vii)  $B$  is nilpotent.

**Lemma 2.5** *Let  $B$  be a block of  $\mathcal{O}G$  for a finite group  $G$  with normal defect group  $D \cong C_2 \times C_2 \times C_2$ . Then  $B$  is Morita equivalent to  $\mathcal{O}(D \rtimes E)$ , where  $E$  has odd order and acts faithfully on  $D$ .*

PROOF. This is well known, but may be obtained for instance by applying Proposition 2.2 and noting that the inertial quotient is one of 1,  $C_3$ ,  $C_7$  and  $C_7 \rtimes C_3$ , each having trivial Schur multiplier.  $\square$

### 3 Preliminary results

**Proposition 3.1** *Let  $N = {}^2G_2(q)$ , where  $q = 3^{2m+1}$  for some  $m \in \mathbb{N} \cup \{0\}$ , and  $N \leq G \leq \text{Aut}(N)$ . Let  $b$  be the principal 2-block of  $\mathcal{O}N$ . Then every block of  $\mathcal{O}G$  covering  $b$  is source algebra equivalent to  $b$ . Further, each of these blocks shares a defect group with  $b$  and has isomorphic inertial quotient.*

PROOF.  $G/N$  is cyclic of odd order. Let  $N = G_0 \leq G_1 \leq \dots \leq G_n = G$ , with each  $|G_{i+1}/G_i|$  prime. By [25]  $b$  has defect groups of the form  $C_2 \times C_2 \times C_2$  and irreducible character degrees occurring with multiplicity either one or two, so that each irreducible character is  $G$ -stable. Since  $[G : N]$  is odd each block of  $\mathcal{O}G_i$  covering  $b$  shares a defect group with  $b$ . By [10], every block with defect group  $C_2 \times C_2 \times C_2$  (in particular  $b$  and every block of  $\mathcal{O}G_i$  covering it) has precisely eight irreducible characters, and it follows that for each  $i$  there are  $[G_{i+1} : N]$  2-blocks of  $\mathcal{O}G_{i+1}$  covering  $b$ , and amongst these there  $[G_{i+1} : G_i]$  blocks of  $\mathcal{O}G_{i+1}$  covering each such block of  $\mathcal{O}G_i$ . It follows from Proposition 2.1 that each block of  $\mathcal{O}G_i$  covering  $b$  is source algebra equivalent to  $b$ . That the blocks have isomorphic inertial quotient follows from [10, 3.4].  $\square$

**Proposition 3.2** *Let  $G$  be a finite group and  $N \triangleleft G$  with  $[G : N]$  an odd prime. Let  $b$  be a  $G$ -stable block of  $\mathcal{O}N$  with defect group  $C_2 \times C_2 \times C_2$  and inertial quotient  $C_3$ . Suppose that  $l(b) = 3$ . Let  $B$  be a block of  $\mathcal{O}G$  covering  $b$ . Then either  $B$  is source algebra equivalent to  $b$  or nilpotent. In the former case  $B$  has inertial quotient  $C_3$  and  $[G : N] = 3$ .*

PROOF. By [10] we have  $l(B) \leq 7$ . Suppose first that  $[G : N] \geq 5$ . Since we are assuming that  $l(b) = 3$ , there cannot be a unique block of  $\mathcal{O}G$  covering  $b$  (since each irreducible Brauer character of  $b$  is  $G$ -stable and so the total number of irreducible Brauer characters in blocks covering  $B$  is at least 15), so by Proposition 2.1  $B$  is source algebra equivalent to  $b$  and has the same inertial quotient.

Suppose now that  $[G : N] = 3$ . If every irreducible Brauer character of  $b$  is  $G$ -stable, in which case again by Proposition 2.1  $B$  is source algebra equivalent to  $b$  and has the same inertial quotient. If the three irreducible Brauer characters are permuted transitively, then  $l(B) = 1$ , so that by [10]  $B$  is nilpotent.  $\square$

The following is a strengthening of a special case of the main result of [11], which is only known to hold for blocks defined over  $k$ .

**Proposition 3.3** *Let  $G$  be a finite group and  $N \triangleleft G$  and let  $C$  be a  $G$ -stable block of  $\mathcal{O}N$  covered by a block  $B$  of  $\mathcal{O}G$  with elementary abelian defect group  $D$  of order 8. Write  $P = N \cap D$  and suppose that  $D = P \times Q$  for some  $Q$  of order 2 such that  $G = N \rtimes Q$ . Then  $B \cong C \otimes_{\mathcal{O}} \mathcal{O}Q$ . In particular  $B$  and the block  $C \otimes_{\mathcal{O}} \mathcal{O}Q$  of  $\mathcal{O}(N \times Q)$  are Morita equivalent.*

PROOF. Write  $Q = \langle x \rangle$ . As noted in [11]  $B$  and  $C$  share a block idempotent  $e$ , so that  $B$  is a crossed product of  $C$  with  $Q$  and it suffices to find a graded unit of  $Z(B)$  of degree  $x$  and order two. We do this by exploiting the existence of a perfect isometry as shown in [10, 5.1], although we must show that this perfect isometry satisfies additional properties. Part of the proof follows that of [10, 5.1], and we take facts from there without explicit further reference. Note however that for convenience we use a different labeling of the irreducible characters.

Denote by  $E$  the inertial quotient of  $B$ , so that  $|E| = 1$  or  $3$ . If  $|E| = 1$ , then  $B$  is nilpotent and the result follows from [22]. Hence we may assume that  $|E| = 3$  and  $E$  acts faithfully on  $D$ . Write  $H = D \rtimes E$ . Then  $Q \leq Z(H)$  and so  $H = (P \rtimes E) \times Q$ .

By [17] we have  $k(B) = 8$ . Label the irreducible characters  $\theta_i$  of  $H$  so that  $\theta_1, \dots, \theta_4$  have  $Q$  in their kernel,  $\theta_1(1) = \theta_2(1) = \theta_3(1) = 1$ ,  $\theta_4(1) = 3$  and  $\theta_i(g) = \theta_{i-4}(g)$  for all  $i = 5, \dots, 8$  and all  $g \in P \rtimes E$ . We have  $\theta_i(x) = -\theta_i(1)$  for  $i = 5, \dots, 8$ . Similarly label the irreducible characters  $\chi_1, \dots, \chi_8$  of  $B$  so that  $\text{Res}_N^G(\chi_i) = \text{Res}_N^G(\chi_{i-4})$  for all  $i = 5, \dots, 8$ . Note that  $\chi_i(x) = -\chi_{i-4}(x)$  for all  $i = 5, \dots, 8$ .

There is a stable equivalence of Morita type between  $\mathcal{O}H$  and  $B$ , leading to an isometry  $L^0(H, \mathcal{O}H) \cong L^0(G, B)$  between the groups of generalised characters vanishing on 2-regular elements.  $L^0(H, \mathcal{O}H)$  is generated by

$$\{\theta_1 - \theta_5, \theta_2 - \theta_6, \theta_3 - \theta_7, \theta_4 - \theta_8, \theta_1 + \theta_2 + \theta_3 - \theta_4\}.$$

We claim that if  $\chi_i - \chi_j \in L^0(G, B)$ , then  $|i - j| = 4$ . For suppose that  $\chi_i(g) = \chi_j(g)$  for all  $g \in G$  of odd order. Then  $\text{Res}_N^G(\chi_i)$  and  $\text{Res}_N^G(\chi_2)$  are irreducible characters of  $C$  agreeing on 2-regular elements. Noting that  $C$  is not nilpotent, and that  $C$  has decomposition matrix that of the principal 2-block of  $A_4$  or  $A_5$ , it follows that  $\text{Res}_N^G(\chi_i) = \text{Res}_N^G(\chi_2)$  and the claim follows.

Hence the isometry takes elements of the form  $\theta_i - \theta_{i-4}$  to elements of the form  $\delta_j(\chi_j - \chi_{j-4})$ . Now the isometry extends to a perfect isometry  $I : \mathbb{Z} \text{Irr}(H) \rightarrow \mathbb{Z} \text{Irr}(B)$ , and we have seen that  $I(\theta_i)(g) = I(\theta_{i-4})(g)$  for every  $i = 5, \dots, 8$  and every  $g \in N$ .

Following [3]  $I$  induces an  $\mathcal{O}$ -algebra isomorphism  $I^0 : Z(\mathcal{O}H) \rightarrow Z(B)$  with  $I^0(x) = \frac{1}{|H|} \sum_{g \in G} \mu(g^{-1}, x)g$ , where  $\mu(g, h) = \sum_{i=1}^8 \theta_i(h)I(\theta_i)(g)$  for  $g \in G$  and  $h \in H$ . We will show that  $I^0(x) = ax$  for some  $a \in \mathcal{O}N$ , i.e., that  $\mu(g, x) = 0$  whenever  $g \in N$ . Then  $I^0(x)$  will be the required graded unit of  $Z(B)$  of degree  $x$  and order two.

Let  $g \in N$ . Then

$$\mu(g, x) = \sum_{i=1}^8 \theta_i(x) I(\theta_i)(g) = \sum_{i=5}^8 \theta_{i-4}(1) (I(\theta_{i-4})(g) - I(\theta_i)(g)) = 0$$

and we are done.  $\square$

## 4 Proof of the main theorem

We prove Theorem 1.1.

PROOF. Let  $B$  be a block of  $\mathcal{O}G$  for a finite group  $G$  with defect group  $D \cong C_2 \times C_2 \times C_2$  with  $[G : Z(G)]$  minimised such that  $B$  is not Morita equivalent to any of (i)-(viii). By minimality and the first Fong reduction  $B$  is quasiprimitive, that is, for every  $N \triangleleft G$  each block of  $\mathcal{O}N$  covered by  $B$  is  $G$ -stable. By Proposition 2.2 if  $N \triangleleft G$  and  $B$  covers a nilpotent block of  $\mathcal{O}N$ , then  $N \leq Z(G)O_2(G)$ . In particular  $O_2(G) \leq Z(G)$ .

Following [2] write  $E(G)$  for the *layer* of  $G$ , that is, the central product of the subnormal quasisimple subgroups of  $G$  (the *components*). Write  $F(G)$  for the Fitting subgroup, which in our case is  $F(G) = Z(G)O_2(G)$ . Write  $F^*(G) = F(G)E(G) \triangleleft G$ , the generalised Fitting subgroup, and note that  $C_G(F^*(G)) \leq F^*(G)$ . Let  $b$  be the (unique) block of  $\mathcal{O}F^*(G)$  covered by  $B$ .

Let  $\overline{B}$  be the unique block of  $\mathcal{O}(G/O_2(Z(G)))$  corresponding to  $B$ . First observe that  $|O_2(Z(G))| \leq 2$ , for otherwise  $\overline{B}$  would have defect at most one and so would be nilpotent, which in turn would mean that  $B$  would be nilpotent by Proposition 2.3, a contradiction.

If  $|O_2(G)| > 4$ , then  $O_2(G) = D$ , a contradiction by Lemma 2.5. Hence  $|O_2(G)| \leq 4$ .

*Claim.*  $O_2(G) \leq Z(G)$  and  $|O_2(G)| \leq 2$ .

Suppose that  $O_2(G) \not\leq Z(G)$  (so  $|O_2(G)| = 4$ ). If  $O_2(Z(G)) \neq 1$ , then  $O_2(G/O_2(Z(G)))$  has order 2 and so is central in  $G/O_2(Z(G))$ , from which it follows using Proposition 2.3 that  $\overline{B}$ , and so  $B$ , is nilpotent, again a contradiction. If  $O_2(Z(G)) = 1$ , then  $F^*(G) = O_2(G) \times (Z(G)E(G))$ . Since  $|O_2(G)| = 4$ ,  $B$  covers a nilpotent block of  $F^*(G)$  and so  $F^*(G) = O_2(G)Z(G)$ . But  $C_G(F^*(G)) \leq F^*(G)$  and so  $D \leq C_G(O_2(G)) \leq O_2(G)Z(G)$ , a contradiction. Hence  $O_2(G) \leq Z(G)$  (and  $|O_2(G)| \leq 2$ ) as claimed.

Write  $E(G) = L_1 * \cdots * L_t$ , where each  $L_i$  is a component of  $G$  (arguing as above we have that  $t \geq 1$ ). Now  $B$  covers a block  $b_E$  of  $\mathcal{O}E(G)$  with defect group contained in  $D$ , and  $b_E$  covers a block  $b_i$  of  $\mathcal{O}L_i$ . Since  $b_E$  is  $G$ -stable, for each  $i$  either  $L_i \triangleleft G$  or  $L_i$  is in a  $G$ -orbit in which each corresponding  $b_i$  is isomorphic (with equal defect). Since  $B$  has defect three, it follows that if  $t > 1$ , then  $B$  covers a nilpotent block of a normal subgroup generated by components of  $G$ , a contradiction. Hence  $t = 1$ . So  $G$  has a unique component  $L_1$ , and  $G/Z(G) \leq \text{Aut}(L_1 Z(G)/Z(G))$ .

Suppose that  $O_2(G) \not\leq [L_1, L_1]$ . Then  $F^*(G) = O_2(G) \times Z(G)L_1$ . In this case  $D \leq F^*(G)$ , since otherwise  $b$  would be nilpotent. Since  $b$  is  $G$ -stable, this means  $[G : F^*(G)]$  odd and so  $O_2(G)$  is in fact a direct factor of  $G$ . By [19] it follows that  $B$  is Morita equivalent to one of (ii) or (iii), a contradiction. Hence  $O_2(G) \leq [L_1, L_1]$ .

We next show that  $D \leq F^*(G)$ . Suppose otherwise. Then since  $D$  is elementary abelian we may write  $D = (D \cap F^*(G)) \times Q$  for some  $Q$  of order 2 (if  $Q$  were to be larger, then  $B$  would cover a nilpotent block of  $\mathcal{O}F^*(G)$ ). By the Schreier conjecture  $G/F^*(G)$  is solvable. Since  $b$  is  $G$ -stable,  $DF^*(G)/F^*(G)$  is a Sylow 2-subgroup of  $G/F^*(G)$ . Hence  $G = H \rtimes Q$  for some  $H \triangleleft G$ . By Proposition 3.3  $B \cong b \otimes_{\mathcal{O}} \mathcal{O}Q$  as  $\mathcal{O}$ -algebras. Now  $b \otimes_{\mathcal{O}} \mathcal{O}Q$  is a block of  $\mathcal{O}(H \times Q)$  with defect group  $D = (D \cap H) \times Q$ . Since  $b$  is Morita equivalent to the principal block of  $\mathcal{O}A_4$  or  $\mathcal{O}A_5$ , it follows that  $B$  is Morita equivalent to one of (ii) or (iii). Hence  $D \leq F^*(G)$ . Since  $[F^*(G) : L_1]$  is odd, this means  $D$  is also a defect group for  $b_1$ .

We now refer to Proposition 2.4. Suppose that  $L_1 \cong SL_2(8)$  and  $b_1$  is the principal block. Then  $G$  is  $SL_2(8)$  or  $\text{Aut}(SL_2(8)) \cong SL_2(8) \rtimes C_3 \cong {}^2G_2(3)$ , leading to (v) or (viii) of the theorem.

If  $L_1 \cong {}^2G_2(3^{2m+1})$  for some  $m \in \mathbb{N}$ , then  $L_1 \leq G \leq \text{Aut}({}^2G_2(3^{2m+1}))$  and by Proposition 3.1  $B$  is Morita equivalent to  $b_1$ . By [21, Example 3.3]  $b_1$  is Morita equivalent to the principal block of  $\mathcal{O}({}^2G_2(3))$ .

If  $L_1 \cong J_1$  or  $Co_3$ , then  $G = L_1$ . By [13, 1.5] the 2-block of  $\mathcal{O}Co_3$  of defect three is Morita equivalent to the principal block of  $\mathcal{O}({}^2G_2(3))$ , hence we are done in this case. The principal block of  $\mathcal{O}J_1$  is not Morita equivalent to that of  $\mathcal{O}({}^2G_2(3))$ , since their decomposition matrices are not similar (see [9] for decomposition matrices).

Suppose that  $L_1$  is of type  $D_n(q)$  or  $E_7(q)$  and  $b_1$  is Morita equivalent to the principal block of  $\mathcal{O}(C_2 \times A_4)$  or  $\mathcal{O}(C_2 \times A_5)$ . Then  $G/L_1$  is abelian and of odd order. By Proposition 3.2  $B$  is either nilpotent (a contradiction) or Morita equivalent to  $b_1$ , and we are done in this case.

This leaves the case that  $|O_2(L_1)| = 2$  and  $D/O_2(L_1)$  is a Klein four group. We have shown that  $O_2(N) = O_2(G)$ . Recall that  $\overline{B}$  is the unique block of  $\mathcal{O}(G/O_2(G))$  corresponding  $B$ , and note that  $\overline{B}$  has defect group  $D/O_2(G)$ . By [4]  $\overline{B}$  is source algebra equivalent to the principal block of  $\mathcal{O}A_4$  or of  $\mathcal{O}A_5$ . It follows from [23, Corollary 1.14] that  $B$  is Morita equivalent to the principal block of a central extension of  $A_4$  or  $A_5$  by a group of order 2, i.e., of  $C_2 \times A_4$  or  $C_2 \times A_5$ .

To see that the blocks in cases (i)-(viii) represent distinct Morita equivalence classes it suffices to note that they have distinct decomposition matrices. □

## ACKNOWLEDGEMENTS

I thank Markus Linckelmann for some useful correspondence on the subject of lifting Morita equivalences.

## References

- [1] J. L. Alperin, *Projective modules for  $SL(2, 2^n)$* , J. Pure and Applied Algebra **15** (1979), 219-234.
- [2] M. Aschbacher, *Finite group theory*, Cambridge Studies in Advanced Mathematics **10**, Cambridge university Press (1986).

- [3] M. Broué, *Isometries parfaites, types de blocs, catégories dérivées*, Astérisque **181-182** (1990), 61–91.
- [4] D. A. Craven, C. W. Eaton, R. Kessar and M. Linckelmann, *The structure of blocks with a Klein four defect group*, Math. Z. **268** (2011), 441–476.
- [5] E. C. Dade, *Block extensions*, Ill. J. Math. **17** (1973), 198–272.
- [6] C. W. Eaton, R. Kessar, B. Külshammer and B. Sambale, *2-blocks with abelian defect groups*, Adv. Math. **254** (2014), 706–735.
- [7] C. W. Eaton, B. Külshammer and B. Sambale, *2-blocks with minimal nonabelian defect groups, II*, J. Group Theory **15** (2012), 311–321.
- [8] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics **1428**, Springer-Verlag (1990).
- [9] The GAP group, GAP - Groups, algorithms, and programming, version 4.4, <http://www.gap-system.org>, 2005.
- [10] R. Kessar, S. Koshitani and M. Linckelmann, *Conjectures of Alperin and Broué for 2-blocks with elementary abelian defect groups of order 8*, J. Reine Angew. Math. **671** (2012), 85–130.
- [11] S. Koshitani and B. Külshammer, *A splitting theorem for blocks*, Osaka J. Math. **33** (1996), 343–346.
- [12] S. Koshitani and H. Miyachi, *donovan conjecture and Loewy length for principal 3-blocks of finite groups with elementary abelian 3-subgroup of order 9*, Comm. Alg. **29** (2001), 4509–4522.
- [13] S. Koshitani, J. Müller and F. Noeske, *Broué’s abelian defect group conjecture holds for the sporadic simple Conway group  $Co_3$* , J. Algebra **358**, 354–380.
- [14] B. Külshammer, *On the structure of block ideals in group algebras of finite groups*, Comm. Alg. **8** (1980), 1867–1872.
- [15] B. Külshammer, *Morita equivalent blocks in Clifford theory of finite groups*, Astérisque **181-182** (1990), 209–168.
- [16] B. Külshammer and L. Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), 17–71.
- [17] P. Landrock, *On the number of irreducible characters in a 2-block*, J. Algebra **68** (1981), 426–442.
- [18] P. Landrock and G. Michler, *Block structure of the smallest Janko group*, Math. Ann. **232** (1978), 205–238.
- [19] M. Linckelmann, *The source algebras of blocks with a Klein four defect group*, J. Algebra **167** (1994), 821–854.



- [20] M. Linckelmann, *The isomorphism problem for cyclic blocks and their source algebras*, Invent. Math. **125** (1996), 265–283.
- [21] , T. Okuyama, *Some examples of derived equivalent blocks of finite group*, preprint (1997).
- [22] L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.
- [23] L. Puig, *Source algebras of  $p$ -central group extensions*, J. Algebra **235** (2001), 359–398.
- [24] B. Sambale, *2-blocks with minimal nonabelian defect groups*, J. Algebra **337** (2011), 261–284.
- [25] H. N. Ward, *On Ree’s series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89.
- [26] A. Watanabe, *On nilpotent blocks of finite groups*, J. Algebra **163** (1994), 128–134.

Charles Eaton  
 School of Mathematics  
 University of Manchester  
 Oxford Road  
 Manchester  
 M13 9PL  
 United Kingdom  
 charles.eaton@manchester.ac.uk